1 Alternative Proof for 3.57

Show that $f(X) = X^{-1}$ is matrix convex on $S^n_{++}$

Proof with the matrix fractional function: $f(x, Y) = x, Y^{-1}x$ is convex on domain $R^n \times S^n_{++}$.

This is shown through the epigraph of $f$:

$\text{epi} f = \{(x, Y, t) | Y \succ 0, x^TY^{-1}x \leq t\} = \{(x, Y, t) | Y \succ 0, \begin{bmatrix} Y & x \\ x^T & t \end{bmatrix} \succeq 0\}$

Where we used that: If $Y \succ 0$, then $X \succeq 0$ iff $S \succeq 0$ and the definition: $X = \begin{bmatrix} Y & x \\ x^T & t \end{bmatrix}$

and $S = t - x^TYx \geq 0$, which is given by the Schur complement condition for pos. semi-def. block matrices.

The condition $\begin{bmatrix} Y & x \\ x^T & t \end{bmatrix} \succeq 0$ is an LMI (linear matrix inequality) in $(x, Y, t)$.

The solution set of an LMI: $\{x | A(x) \preceq B\}$ is convex, since it is the inverse image of the pos. semindef. cone under the affine fct. $f(x) = B - A(x)$

$S^n_{++} = \{X \in S^n, X \succeq 0\}$
$f^{-1}(S^n_{++}) = \{x \in R^n | B - A(x) \succeq 0\}$